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Robust H_∞ rotating consensus control for second-order multi-agent systems with uncertainty and time-varying delay in three-dimensional space[1]

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Abstract. The robust delay-dependent H_∞ control problems for rotating consensus of second-order multi-agent systems is studied, which is subject to uncertainty, external disturbances and time-varying delay in three-dimensional space. First, a rotating consensus is defined in three-dimensional space. Then a distributed control protocol based on state feedback of neighbors is designed. In the next place, a sufficient delay-dependent condition in terms of the matrix inequalities is derived to make all agents asymptotically reach rotating consensus with the desired H_∞ performance index. Furthermore, an algorithm is elaborately designed to get a feasible solution to this condition. Finally, simulation results are provided to illustrate the effectiveness of our theoretical results and algorithms.

Key words. Multi-agent systems, rotating consensus, H_∞ control, uncertainty, time-varying delay.

1. Introduction

As a special case of the consensus problems, the rotating consensus is used to describe a class of collective circular motions such as the motion of celestial bodies, flocks of birds flying around a closed circuit course and schools of fish swimming along an approximately circular orbit, which can find important potential applications in formation flight of satellites around the earth, spacecraft docking, circular mobile sensor networks, etc. However, most of existing results cannot be straightly applied to imitate or explain such motions and rare results are derived to generate such motions. In [1], Sepulchre et al. formulated a new rotating formation control problem

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according to application of autonomous underwater vehicles to collect oceanographic measurements. In [2] and in [3], Lin et al. investigated collective rotating motions of second-order multi-agent systems on a plane and in three dimensional space, respectively. In [4], Yang et al. studied distributed rotating consensus in networks of second-order agents using only local position information in three-dimensional space. In [5], distributed composite rotating consensus problem was investigated for second-order multi-agent systems, where all agents move in a nested circular orbit.

As a typical networked system, multi-agent systems often contain disturbances, uncertainties and time-delay in practical applications. Moreover, the existence of these facts might destroy the convergence properties of the systems. Therefore, it is significant to investigate robust consensus problems of multi-agent systems, which reflect the effects of these facts on their behavior. In the past decade, some interesting results have been obtained for robust consensus problems in [6–9]. However, these results cannot be straight applied to robust delay-dependent H_∞ control problems for rotating consensus [10]. So far, only the author of this paper studied distributed robust H_∞ rotating consensus control problems for directed networks of second-order agents with mixed uncertainties and time-delay in [10], but we only considered constant time delay, rotating consensus on a plane, and we only obtained the sufficient conditions that the parameters of control protocols should be satisfied, but their calculation method was not directly given in [10]. Therefore, this paper will further study robust delay-dependent H_∞ control problems for rotating consensus of second-order multi-agent systems with uncertainty and time-varying delay in three-dimensional space.

2. State of the art

Consider a multi-agent system consisting of n second-order agents. Each agent is regarded as a node in a graph $\mathcal{G}$. Suppose that the ith agent

$$s_i,\ (i \in I_{\mathrm{g}},\, I_{\mathrm{g}} = \{1,\, 2\, \cdots,\, n\})$$

has the dynamics as follows:

$$\begin{aligned} \dot{r}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t) + w_i(t), \end{aligned} \tag{1}$$

where $r_i\,(t)$, $v_i\,(t)$ denote the position and velocity state of the ith agent s_i, $u_i\,(t)$ denote the control input or control protocol of the ith agent, and $w_i\,(t) \in \mathcal{L}_2\,[0,\infty)$ denotes the external disturbances of the ith agent. Finally, $r_i\,(t)$, $v_i\,(t)$, $u_i\,(t)$, $w_i\,(t) \in \mathcal{R}^3$.

In this paper, our main objective is to design a distributed protocol to make all agents reach rotating consensus while satisfying the desired H_∞ disturbance attenuation index. It is called that all agents reach rotating consensus if all agents reach consensus and surround a common point with a desired constant angular velocity $\omega \in \mathcal{R}$ on a plane, whose normal is a specified unit vector $\mathrm{i}_\omega \in \mathcal{R}^3$. The rotating consensus can be defined as follows

Definition 1.[3]: The multi-agent system (1) reaches rotating consensus, if and only if the states of agents satisfy

$$\begin{aligned} &\lim_{t\to+\infty} \mathrm{i}_\omega^{\mathrm{T}} v_i(t) = 0, \\ &\lim_{t\to+\infty} (\dot{v}_i(t) - \omega R_\omega v_i(t)) = 0, \\ &\lim_{t\to+\infty} (v_i(t) - v_k(t)) = 0, \\ &\lim_{t\to+\infty} (c_i(t) - c_k(t)) = 0 \end{aligned} \tag{2}$$

for $\forall i, j \in I_{\mathrm{g}}$, where ω denotes the desired constant angular velocity, $c_i(t) = r_i(t) + \omega^{-1} R_\omega v_i(t)$

$$R_\omega = R_{\omega 0}^{\mathrm{T}} R_{i_\omega}(\pi/2) R_{\omega 0} R_{i_\omega}(\pi/2) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Methodology

In this section, we will solve H_∞ control problem for rotating consensus of multi-agent systems with parameter uncertainties and time-delay in three-dimensional space. Firstly, we will design a distributed control protocol. Then, we will deduce the rule and algorithm for designing state feedback matrix $\boldsymbol{K}$.

3.1. Protocol design

Based on the state feedback of neighbors, the distributed control protocol we used is given as

$$u_i(t) = u_{i1}(t) + u_{i2}(t) \tag{3}$$

for all $i \in I_{\mathrm{g}}$, where

$$\begin{aligned} u_{i1}(t) &= \omega R_\omega v_i(t), \\ u_{i2}(t) &= (\boldsymbol{I}_3 + \Delta\boldsymbol{B}(t))[\boldsymbol{K}_{\mathrm{v}} \textstyle\sum_{s_k\in N_i} a_{ik}(v_k(t-d(t)) - v_i(t-d(t))] + \\ &\quad + \boldsymbol{K}_{\mathrm{c}} \textstyle\sum_{s_k\in N_i} a_{ik}(c_k(t-d(t)) - c_i(t-d(t))). \end{aligned}$$

In the protocol, $\boldsymbol{K}_{\mathrm{v}}$ and $\boldsymbol{K}_{\mathrm{v}}$ are the state feedback matrices, ω denotes the desired constant angular velocity, $d(t)$ $(0 < d(t) \le \tau,\ |\dot{d}(t)| < \mu)$ denotes the time-varying delay, N_i denotes the set of neighbors of agent s_i and $\Delta\boldsymbol{B}(t)$ is a matrix-valued function representing time-varying parameter uncertainties. The parameter uncertainties are assumed to be norm-bounded and $\Delta\boldsymbol{B}(t) = \boldsymbol{G}\boldsymbol{F}(t)\boldsymbol{E}$, where $\boldsymbol{G}$ and $\boldsymbol{E}$

are known constant matrices with appropriate dimensions and $\boldsymbol{F}(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t) \leq \boldsymbol{I}$ for all $t \geq 0$.

Define the output functions $z_i(t)$, which is computed from the average of the relative states of all agents as follows:

$$z_i(t) = [z_{i1}^{\mathrm{T}}(t),\, z_{i2}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{R}^6, \qquad i \in I_{\mathrm{g}}, \tag{4}$$

where $z_{i1}(t) = v_i(t) - \frac{1}{n}\sum\limits_{j=1}^{n} v_j(t)$ and $z_{i1}(t) = c_i(t) - \frac{1}{n}\sum\limits_{j=1}^{n} c_j(t)$.

Therefore, it is clear that the rotating consensus of the system (1) can be reached if and only if the states of agent satisfy

$$\begin{aligned} &\lim_{t\to+\infty} z_i(t) = 0 \\ &\lim_{t\to+\infty} [\dot{v}_i(t) - \omega R_\omega v_i(t)] = 0 \end{aligned} \tag{5}$$

for all $i \in I_{\mathrm{g}}$.

Denote

$$\begin{aligned} &\xi(t) = [\xi_1^{\mathrm{T}}(t), \cdots, \xi_n^{\mathrm{T}}(t)]^{\mathrm{T}}, && \xi_i(t) = [v_i^{\mathrm{T}}(t), c_i^{T}(t)]^{\mathrm{T}}, \\ &w(t) = [w_1^{\mathrm{T}}(t), \cdots, w_n^{\mathrm{T}}(t)]^{\mathrm{T}}, && z(t) = [z_1^{\mathrm{T}}(t), \cdots, z_n^{\mathrm{T}}(t)]^{\mathrm{T}}, \\ &\boldsymbol{A} = \begin{bmatrix} \omega R_\omega & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}, && \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{I}_3 \\ \omega^{-1} R_\omega \end{bmatrix}, \\ &\boldsymbol{K} = [\boldsymbol{K}_v, \boldsymbol{K}_c], && \Delta\boldsymbol{B}_1(t) = \boldsymbol{B}_1 \Delta\boldsymbol{B}(t), \\ &\boldsymbol{C} = [C_{ij}]_{i,j=1}^{n}, && C_{ij} = \begin{cases} \frac{n-1}{n}, & i = j, \\ -\frac{1}{n}, & i \neq j. \end{cases} \end{aligned}$$

By using the protocol (3), the closed-loop dynamics of the system (1) can be written as

$$\begin{aligned} &\dot{\xi}(t) = (\boldsymbol{I}_n \otimes \boldsymbol{A})\xi(t) - \boldsymbol{L} \otimes [(\boldsymbol{B}_1 + \Delta\boldsymbol{B}_1(t))\boldsymbol{K}]\xi(t - d(t)) + (\boldsymbol{I}_n \otimes \boldsymbol{B}_2)w(t), \\ &z(t) = (\boldsymbol{C} \otimes \boldsymbol{I}_6)\xi(t), \qquad 0 < d(t) \leq \tau, \quad \dot{d}(t) \leq \mu, \end{aligned} \tag{6}$$

where $\otimes$ denotes the Kronecker product and $\boldsymbol{L}$ is the Laplacian matrix of graph $\mathcal{G}$.

According to robust control theory, the attenuating ability of consensus performance for multi-agent system (1) against external disturbances can be quantitatively measured by the H_∞ norm of the closed-loop transfer function matrix $\boldsymbol{T}_{wz}(s)$ from the external disturbance $w(t)$ to the controlled output $z(t)$, we design a distributed state feedback protocol $u_i(t)$ such that

$$\| \boldsymbol{T}_{wz}(s) \|_\infty < \gamma \tag{7}$$

holds for prescribed H_∞ disturbance attenuation index γ.

3.2. Some necessary lemmas

Lemma 1. [8]: Assume that the interaction graph $\mathcal{G}$ is connected. For a given $\gamma > 0$, the closed-loop system (6) reaches consensus with the desired H_∞ disturbance attenuation index $\gamma\,(\boldsymbol{T}_{wz}(s)_\infty < \gamma)$, if and only if the following $n-1$ systems are simultaneously asymptotically stable with $\boldsymbol{T}_{\hat{w}\hat{z}}(s)_\infty < \gamma$ for $i = 1, 2, \cdots, n-1$.

$$\begin{aligned} \dot{\hat{\delta}}_i(t) &= A\hat{\delta}_i(t) - \lambda_i(\boldsymbol{B}_1 + \Delta\boldsymbol{B}_1(t))\boldsymbol{K}\hat{\delta}_i(t - d(t)) + \boldsymbol{B}_2\hat{w}_i(t), \\ \hat{z}_i(t) &= \hat{\delta}_i t), \quad 0 < d(t) \le \tau,\ \dot{d}(t) \le \mu, \end{aligned} \tag{8}$$

where λ_i, $(i = 1, 2, n-1)$ are positive eigenvalues of the Laplacian matrix $\boldsymbol{L}$, and $\hat{\delta}_i(t),\ \hat{w}_i(t),\ \hat{z}_i(t) \in \mathcal{R}^6$.

Lemma 2. (Schur complement formula) [6]: For a given symmetric matrix $\boldsymbol{S}$ with the form $\boldsymbol{S} = [S_{ij}]$, $S_{11} \in \mathcal{R}^{r\times r}$, $S_{12} \in \mathcal{R}^{r\times(n-r)}$ or $S_{22} \in \mathcal{R}^{(n-r)\times(n-r)}$, then $S < 0$ if and only if $S_{11} < 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$ or $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

Consider a nominal time delay system as follows:

$$\begin{aligned} \dot{x}_{(t)} &= \boldsymbol{A}x(t) + \boldsymbol{A}_d x(t - d(t)) + \boldsymbol{B}_w w(t), \\ z(t) &= \boldsymbol{C}x(t), \quad 0 < d(t) \le \tau,\ \dot{d}(t) \le \mu. \end{aligned} \tag{9}$$

A bounded real lemma (BRL) will be introduced in the following part.

Lemma 3. (BRL) [12]: For a given $\gamma, \tau, \mu > 0$, the nominal time delay system (9) is asymptotically stable with $\boldsymbol{C}_{wz}(s)_\infty < \gamma$, if there exist positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$ and matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ with appropriate dimensions such that

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} & -\boldsymbol{M}_1 & \boldsymbol{P}\boldsymbol{B}_w & \tau A^T\boldsymbol{S} & \tau\boldsymbol{M}_1 & \tau\boldsymbol{N}_1 \\ * & \boldsymbol{\Gamma}_{22} & -\boldsymbol{M}_2 & 0 & \tau\boldsymbol{A}_d^{\boldsymbol{T}}\boldsymbol{S} & \tau\boldsymbol{M}_2 & \tau\boldsymbol{N}_2 \\ * & * & -\boldsymbol{R} & 0 & 0 & 0 & 0 \\ & * & * & -\gamma^2\boldsymbol{I} & \tau\boldsymbol{B}_w^{\mathrm{T}}\boldsymbol{S} & 0 & 0 \\ * & * & * & * & -\tau\boldsymbol{S} & 0 & 0 \\ * & * & * & * & * & -\boldsymbol{S} & 0 \\ * & * & * & * & * & * & -\boldsymbol{S} \end{bmatrix} < 0 \tag{10}$$

where

$$\boldsymbol{\Gamma}_{11} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{R} + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\,,$$

$$\boldsymbol{\Gamma}_{12} = \boldsymbol{P}\boldsymbol{A}_d + \boldsymbol{M}_1 - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}}\,,$$

$$\boldsymbol{\Gamma}_{12} = (\mu - 1)\,\boldsymbol{Q} + \boldsymbol{M}_2 + \boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}}\,.$$

3.3. Condition of robust H_∞ rotating consensus

Theorem 1. Assume that the interaction graph $\mathcal{G}$ is connected. For given positive constants τ, μ and γ, by distributed protocol (3), multi-agent system (1) can reach rotating consensus while satisfying desired H_∞ disturbance attenuation index, if there exist positive definite matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{X} \in \mathcal{R}^{6\times 6}$, matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2 \in \mathcal{R}^{6\times 6}$, $\boldsymbol{Y} \in \mathcal{R}^{3\times 6}$ and a positive scalar ε such that

$$\boldsymbol{\Phi}_i = \begin{bmatrix} \boldsymbol{\Phi}_{i0} & \boldsymbol{\Pi}_{i1}^{\mathrm{T}} & \boldsymbol{\Pi}_2^{\mathrm{T}} & \boldsymbol{H}^{\mathrm{T}} \\ * & -\boldsymbol{T} + \varepsilon\lambda_i^2\tau^2\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} & 0 & 0 \\ * & * & -\boldsymbol{X}\boldsymbol{T}^{-1}\boldsymbol{X} & 0 \\ * & * & * & -\varepsilon\boldsymbol{I} \end{bmatrix} < 0 \tag{11}$$

is satisfied for $i = 1$ and $n-1$, where

$$\boldsymbol{\Phi}_{i0} = \begin{bmatrix} \boldsymbol{\Phi}_{i11} & \boldsymbol{\Phi}_{i12} & -\boldsymbol{M}_1 & \boldsymbol{B}_2 & 0 & \tau\boldsymbol{M}_1 & \tau\boldsymbol{N}_1 & \boldsymbol{X} \\ * & \boldsymbol{\Phi}_{22} & -\boldsymbol{M}_2 & 0 & 0 & \tau\boldsymbol{M}_2 & \tau\boldsymbol{N}_2 & 0 \\ * & * & -\boldsymbol{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2\boldsymbol{I} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\tau\boldsymbol{S} & 0 & 0 & 0 \\ * & * & * & * & * & -\tau\boldsymbol{S} & 0 & 0 \\ * & * & * & * & * & * & -\tau\boldsymbol{S} & 0 \\ * & * & * & * & * & * & * & -\boldsymbol{I} \end{bmatrix},$$

$$\boldsymbol{\Phi}_{i11} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q} + \boldsymbol{R} + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \varepsilon\lambda_i^2\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}},$$

$$\boldsymbol{\Phi}_{i12} = -\lambda_i\boldsymbol{B}_1\boldsymbol{Y} + \boldsymbol{M}_1 - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}},$$

$$\boldsymbol{\Phi}_{22} = (\mu - 1)\boldsymbol{Q} + \boldsymbol{M}_2 + \boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{N}_2 - \boldsymbol{N}_2^{\mathrm{T}},$$

$$\boldsymbol{\Pi}_{i1} = [\tau\boldsymbol{A}\boldsymbol{X} + \varepsilon\lambda_i^2\tau\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}}, -\lambda_i\tau\boldsymbol{B}_1\boldsymbol{Y}, 0, \tau\boldsymbol{B}_2, 0, 0, 0, 0],$$

$$\boldsymbol{\Pi}_2 = [0, 0, 0, 0, \boldsymbol{S}, 0, 0, 0],$$

$$H = [0, \boldsymbol{E}\boldsymbol{Y}, 0, 0, 0, 0, 0, 0].$$

Proof: According to Lemma 3, the subsystem (11) is asymptotically stable with $\boldsymbol{T}_{\hat{w}\hat{z}}(s)_\infty < \gamma$, if there exist positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$ and matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ with appropriate dimensions such that

$$\begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{i12} & -\boldsymbol{M}_1 & \boldsymbol{PB}_2 & \tau\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} & \tau\boldsymbol{M}_1 & \tau\boldsymbol{N}_1 \\ * & \boldsymbol{\Gamma}_{22} & -\boldsymbol{M}_2 & 0 & -\lambda_i\tau(\boldsymbol{B}_d\boldsymbol{K})^{\mathrm{T}}\boldsymbol{S} & \tau\boldsymbol{M}_2 & \tau\boldsymbol{N}_2 \\ * & * & -\boldsymbol{R} & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2\boldsymbol{I} & \tau\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S} & 0 & 0 \\ * & * & * & * & -\tau\boldsymbol{S} & 0 & 0 \\ * & * & * & * & * & -S & 0 \\ * & * & * & * & * & * & -\boldsymbol{S} \end{bmatrix} \triangleq \boldsymbol{\Gamma}_i < 0, \quad (12)$$

where

$$\boldsymbol{B}_d = \boldsymbol{B}_1 + \Delta\boldsymbol{B}_1(t)\ ,$$
$$\boldsymbol{\Gamma}_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{R} + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{I}\ ,$$
$$\boldsymbol{\Gamma}_{i12} = -\lambda_i\boldsymbol{PB}_d\boldsymbol{K} + \boldsymbol{M}_1 - \boldsymbol{N}_1 + \boldsymbol{N}_2^{\mathrm{T}}\ ,$$
$$\boldsymbol{\Gamma}_{22} = (\mu - 1)\,\boldsymbol{Q} + \boldsymbol{M}_2 + \boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{N}_2 + \boldsymbol{N}_2^{\mathrm{T}}\ .$$

Due to the convex property of linear matrix inequality (LMI) of $\boldsymbol{\Gamma}_i$, $\boldsymbol{\Gamma}_i < 0$ for all $i = 1,\ 2,\ n-1$ if and only if $\boldsymbol{\Gamma}_1 < 0$ and $\boldsymbol{\Gamma}_{n-1} < 0$, which is associated with the smallest eigenvalues λ_1 and the largest eigenvalues λ_{n-1}, respectively.

Pre- and post-multiplying the inequality (12) with

$$\mathrm{diag}\left\{\boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{I}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}\right\}$$

and applying the variable changes $\boldsymbol{X} = \boldsymbol{P}^{-1}$, $\hat{*} = \boldsymbol{P}^{-1} * \boldsymbol{P}^{-1}$, where $*$ denotes $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S}, \boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{N}_1, \boldsymbol{N}_2$, the inequality (12) is congruent to $\hat{\boldsymbol{\Gamma}}_{i0} + \hat{\boldsymbol{\Gamma}}_{i1} + \hat{\boldsymbol{\Gamma}}_{i1}^{\mathrm{T}} < 0$. On one hand, $\hat{\boldsymbol{\Gamma}}_{i1}$ can be rewritten as $\hat{\boldsymbol{\Gamma}}_{i1} = \hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{X}^{-1}\hat{\boldsymbol{\Pi}}_2$, where $\hat{\boldsymbol{\Pi}}_2 = \left[0, 0, 0, 0, \hat{\boldsymbol{S}}, 0, 0\right]$ and $\hat{\boldsymbol{\Pi}}_{i1} = [\tau\boldsymbol{AX}, -\lambda_i\tau\boldsymbol{B}_d\boldsymbol{KX}, 0, \tau\boldsymbol{B}_2, 0, 0, 0]$. On the other hand, according to square inequality, it is easy to construct the following inequality:

$$\hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{X}^{-1}\hat{\boldsymbol{\Pi}}_2 + (\hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{X}^{-1}\hat{\boldsymbol{\Pi}}_2)^{\mathrm{T}} \le \hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{T}^{-1}\hat{\boldsymbol{\Pi}}_{i1} + \hat{\boldsymbol{\Pi}}_2^{\mathrm{T}}\boldsymbol{X}^{-1}\boldsymbol{T}\boldsymbol{X}^{-1}\hat{\boldsymbol{\Pi}}_2\ ,$$

where $\boldsymbol{T}$ is any symmetric positive definite matrix with appropriate dimensions. Therefore, $\hat{\boldsymbol{\Gamma}}_{i0} + \hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{T}^{-1}\hat{\boldsymbol{\Pi}}_{i1} + \hat{\boldsymbol{\Pi}}_2^{\mathrm{T}}\boldsymbol{X}^{-1}\mathrm{T}\boldsymbol{X}^{-1} < 0$ implies that $\hat{\boldsymbol{\Gamma}}_i < 0$. Then, defining $\boldsymbol{Y} = \boldsymbol{KX}$ and applying Lemma 2 (Schur complement formula) on $\hat{\boldsymbol{\Gamma}}_{i0} + \hat{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}}\boldsymbol{T}^{-1}\hat{\boldsymbol{\Pi}}_{i1} + \hat{\boldsymbol{\Pi}}_2^{\mathrm{T}}\boldsymbol{X}^{-1}\boldsymbol{T}\boldsymbol{X}^{-1} < 0$, we obtain the matrix inequality condition

$$\bar{\boldsymbol{\Phi}}_i = \begin{bmatrix} \bar{\boldsymbol{\Phi}}_{i0} & \bar{\boldsymbol{\Pi}}_{i1}^{\mathrm{T}} & \bar{\boldsymbol{\Pi}}_2^{\mathrm{T}} \\ * & -\boldsymbol{T} & 0 \\ * & * & -\boldsymbol{X}\boldsymbol{T}^{-1}\boldsymbol{X}\ , \end{bmatrix} < 0 \quad (13)$$

where

$$\bar{\boldsymbol{\Pi}}_{i1} = [\tau\boldsymbol{AX}, -\lambda_i\boldsymbol{B}_d\boldsymbol{KX}, 0, \tau\boldsymbol{B}_2, 0, 0, 0, 0],$$
$$\bar{\boldsymbol{\Pi}}_2 = [0, 0, 0, 0, \hat{\boldsymbol{S}}, 0, 0, 0] \quad ,$$

$$\bar{\boldsymbol{\Pi}}_{i0}=\begin{bmatrix}\bar{\boldsymbol{\Pi}}_{11} & \bar{\boldsymbol{\Pi}}_{i12} & -\hat{\boldsymbol{M}}_1 & \boldsymbol{B}_2 & 0 & \tau\hat{\boldsymbol{M}}_1 & \tau\hat{\boldsymbol{N}}_1 & \boldsymbol{X}\\ * & \boldsymbol{\Pi}_{22} & -\hat{\boldsymbol{M}}_2 & 0 & 0 & \tau\hat{\boldsymbol{M}}_2 & \tau\hat{\boldsymbol{N}}_2 & 0\\ * & * & -\hat{\boldsymbol{R}} & 0 & 0 & 0 & 0 & 0\\ * & * & * & -\gamma^2\boldsymbol{I} & 0 & 0 & 0 & 0\\ * & * & * & * & -\tau\hat{\boldsymbol{S}} & 0 & 0 & 0\\ * & * & * & * & * & -\hat{\boldsymbol{S}} & 0 & 0\\ * & * & * & * & * & * & -\hat{\boldsymbol{S}} & 0\\ * & * & * & * & * & * & * & -\boldsymbol{I}\end{bmatrix},$$

$$\bar{\boldsymbol{\Pi}}_{11}=\boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\hat{\boldsymbol{Q}}+\hat{\boldsymbol{R}}+\hat{\boldsymbol{N}}_1+\hat{\boldsymbol{N}}_1^{\mathrm{T}},$$

$$\bar{\boldsymbol{\Pi}}_{i12}=-\lambda_i\boldsymbol{B}_d\boldsymbol{Y}+\hat{\boldsymbol{M}}_1-\hat{\boldsymbol{N}}_1+\hat{\boldsymbol{N}}_2^{\mathrm{T}},$$

$$\bar{\boldsymbol{\Pi}}_{22}=(\mu-1)\hat{\boldsymbol{Q}}+\hat{\boldsymbol{M}}_2+\hat{\boldsymbol{M}}_2^{\mathrm{T}}-\hat{\boldsymbol{N}}_1-\hat{\boldsymbol{N}}_1^{\mathrm{T}}.$$

Note that $\boldsymbol{B}_d=\boldsymbol{B}_1+\Delta\boldsymbol{B}_1(t)=\boldsymbol{B}_1+\boldsymbol{B}_1\boldsymbol{GF}(t)\boldsymbol{E}$. Define

$$\begin{aligned}\boldsymbol{J}_i &\triangleq [-\lambda_i(\boldsymbol{B}_1\boldsymbol{G})^{\mathrm{T}},0,0,0,0,0,0,0,-\lambda_i\tau(\boldsymbol{B}_1\boldsymbol{G})^{\mathrm{T}},0]^{\mathrm{T}},\\ \boldsymbol{H} &\triangleq [0,\boldsymbol{EY},0,0,0,0,0,0,0,0].\end{aligned}$$

Now we can obtain $\bar{\boldsymbol{\Phi}}_i=\hat{\boldsymbol{\Phi}}_i+\boldsymbol{J}_i\boldsymbol{F}(t)\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{J}_i^{\mathrm{T}}$, where $\hat{\boldsymbol{\Phi}}_i$ is constructed through replacing $\boldsymbol{B}_d$ term in $\bar{\boldsymbol{\Phi}}_i$ with $\boldsymbol{B}_1$. Note that $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\leq\boldsymbol{I}$, so that there exists an $\varepsilon>0$ such that $\hat{\boldsymbol{\Phi}}_i+\varepsilon\boldsymbol{J}_i\boldsymbol{J}_i^{\mathrm{T}}+\varepsilon^{-1}\boldsymbol{H}^T\boldsymbol{H}<0$ guaranteeing $\bar{\boldsymbol{\Phi}}_i<0$. Therefore, applying Lemma 2 on $\hat{\boldsymbol{\Phi}}_i+\varepsilon\boldsymbol{J}_i\boldsymbol{J}_i^{\mathrm{T}}+\varepsilon^{-1}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}<0$ and using the definitions $*\triangleq\hat{*}$, where $*$ denotes $\boldsymbol{Q},\boldsymbol{R},\boldsymbol{S},\boldsymbol{M}_1,\boldsymbol{M}_2,N_1,\boldsymbol{N}_2$, we can obtain that $\boldsymbol{\Phi}_i<0$ (see (11)) and $\boldsymbol{K}=\boldsymbol{YX}^{-1}$.

On the basis of the above analysis, if $\boldsymbol{\Phi}_1<0$, $\boldsymbol{\Phi}_{n-1}<0$ and $\boldsymbol{K}=\boldsymbol{YX}^{-1}$, the subsystems (9) are simultaneously asymptotically stable with $\boldsymbol{T}_{\hat{w}\hat{z}}(s)_\infty<\gamma$. Then, by Lemma 1, the multi-agent system (1) reaches rotating consensus while satisfying desired H_∞ disturbance with attenuation index γ. This completes the proof.

3.4. Algorithm for robust H_∞ rotating consensus

Because the matrix inequality condition (11) is not in the form of LMI, we cannot directly use the LMI method to solve the matrix inequality (11). But we can turn this problem into the LMI optimization problem by the cone-complementary linearization algorithm [13]. First, we define a new variable $\boldsymbol{U}=\boldsymbol{U}^{\mathrm{T}}>0$ such that $\boldsymbol{U}\leq\boldsymbol{XT}^{-1}\boldsymbol{X}$. It is easy to derive that $\boldsymbol{U}^{-1}-\boldsymbol{X}^{-1}\boldsymbol{TX}^{-1}\geq 0$. Furthermore, by defining $\bar{\boldsymbol{X}}\triangleq\boldsymbol{X}^{-1}$, $\bar{\boldsymbol{T}}\triangleq\boldsymbol{T}^{-1}$, $\bar{\boldsymbol{U}}\triangleq\boldsymbol{U}^{-1}$, and using Lemma 2, we can turn the condition (11) into

$$\begin{bmatrix}\boldsymbol{\Phi}_{i0} & \boldsymbol{\Pi}_{i1}^{\mathrm{T}} & \boldsymbol{\Pi}_2^{\mathrm{T}} & \boldsymbol{H}^{\mathrm{T}}\\ * & -\boldsymbol{T}+\varepsilon\lambda_i^2\tau^4\boldsymbol{B}_1\boldsymbol{GG}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} & 0 & 0\\ * & * & \boldsymbol{U} & 0\\ * & * & * & -\varepsilon\boldsymbol{I}\end{bmatrix}<0 \tag{14}$$

and

$$\begin{bmatrix} \bar{\boldsymbol{U}} & \bar{\boldsymbol{X}} \\ \bar{\boldsymbol{X}} & \bar{\boldsymbol{T}} \end{bmatrix} \geq 0, \begin{bmatrix} \bar{\boldsymbol{X}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X} \end{bmatrix} \geq 0, \begin{bmatrix} \bar{\boldsymbol{T}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{T} \end{bmatrix} \geq 0, \begin{bmatrix} \bar{U} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{U} \end{bmatrix} \geq 0. \tag{15}$$

Then, we may solve the following optimization problem and find a feasible solution satisfying $\boldsymbol{U} \leq \boldsymbol{X}\boldsymbol{T}^{-1}\boldsymbol{X}$:

$$\begin{array}{ll} \min & \mathrm{Trace}(\bar{\boldsymbol{X}}\boldsymbol{X} + \bar{\boldsymbol{T}}\boldsymbol{T} + \bar{\boldsymbol{U}}\boldsymbol{U}) \\ \text{s.t.} & (14), (15). \end{array} \tag{16}$$

In conclusion, in order to solve matrix inequality condition (11), an algorithm is designed as follows:

Algorithm 1:

Step 1. Solve the LMIs (14) and (15) for given positive scalar constants τ, μ and γ. There exists a feasible solution set $\left\{\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0, \bar{\boldsymbol{T}}_0, \boldsymbol{T}_0, \bar{\boldsymbol{U}}_0, \boldsymbol{U}_{0,}\right\}$ and set $k = 0$.

Step 2. Solve the following optimization problem for the variables

$$\begin{array}{ll} & \{\bar{\boldsymbol{X}}, \boldsymbol{X}, \bar{\boldsymbol{T}}, \boldsymbol{T}, \bar{\boldsymbol{U}}, \boldsymbol{U}\}, \\ \min & \mathrm{Trace}(\bar{\boldsymbol{X}}_k\boldsymbol{X} + \bar{\boldsymbol{T}}_k\boldsymbol{T} + \bar{\boldsymbol{U}}_k\boldsymbol{U} + \bar{\boldsymbol{X}}\boldsymbol{X}_k + \bar{\boldsymbol{T}}\boldsymbol{T}_k + \bar{\boldsymbol{U}}\boldsymbol{U}_k), \\ \text{s.t.} & (14),\ (15) \end{array} \tag{17}$$

and set $\bar{\boldsymbol{X}}_{k+1} = \bar{\boldsymbol{X}}$, $\boldsymbol{X}_{k+1} = \boldsymbol{X}$, $\bar{\boldsymbol{T}}_{k+1} = \bar{\boldsymbol{T}}$, $\boldsymbol{T}_{k+1} = \boldsymbol{T}$, $\bar{\boldsymbol{U}}_{k+1} = \bar{\boldsymbol{U}}$, $\boldsymbol{U}_{k+1} = \boldsymbol{U}$.

Step 3. If $\boldsymbol{U} \leq \boldsymbol{X}\boldsymbol{T}^{-1}\boldsymbol{X}$ for the above solution set, then save the current $\boldsymbol{X}$, $\boldsymbol{Y}$ and exit. Otherwise, set $k = k + 1$, go to step 2 and repeat the optimization for a prescribed maximum iterative number $k_{\max}$ until finding a feasible solution satisfying $\boldsymbol{U} \leq \boldsymbol{X}\boldsymbol{T}^{-1}\boldsymbol{X}$. If such a solution does not exist, then exit.

If a feasible solution set is found by Algorithm 1 for the given disturbance attenuation index γ, delay parameter τ and μ by distributed protocol (3), the multi-agent system (1) can reach rotating consensus with the desired H_∞ disturbance attenuation index γ and the feedback matrix can be constructed by $\boldsymbol{K} = [\boldsymbol{K}_v\ \boldsymbol{K}_c] = \boldsymbol{Y}\boldsymbol{X}^{-1}$.

4. Result analysis and discussion

To illustrate the obtained theoretical results and optimization algorithms, numerical simulations will be given in this section. Figure 1 shows a communication topology for the multi-agent system (1) when $n = 4$.

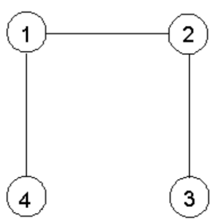

Fig. 1. Communication topology of four-agent system

Suppose that the weight of each edge is 1, the desired angular velocity $\omega = 1$ and the normal vector of the desired rotating plane $\mathrm{i}_\omega = \begin{bmatrix} -0.5 & -0.5 & 0.5\sqrt{2} \end{bmatrix}^{\mathrm{T}}$.In order to clearly reflect the effect of external disturbances to the rotating consensus performance, we first assume the external disturbance

$$w(t) = [0.4, 0.1, 0.9, 0.2, 0, 0.4, -0.5, -0.3, -0.8, -0.5, -0.3, -1.3]^{\mathrm{T}}$$

where $\varepsilon(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$.

Secondly, we suppose that the parameter uncertainty matrix $\Delta \boldsymbol{B}(t) = \boldsymbol{G}\boldsymbol{F}(t)\boldsymbol{E}$, where $\boldsymbol{G} = [0.02, 0.01, 0; 0.01, 0.02, 0; 0, 0, 0.01]$, $\boldsymbol{F}(t) = \mathrm{diag}\{\sin 10t, \sin 20t, \cos 20t\}$ and $\boldsymbol{E} = \boldsymbol{I}_3$. At last, we presume that the initial state of the multi-agent system is taken as $\xi(t=0) = 1_4 \otimes \left[v_0^{\mathrm{T}}, c_0^{\mathrm{T}}\right]^{\mathrm{T}}$, where $v_0 = R_{\omega 0}^{\mathrm{T}}[0, -1, 0]^{\mathrm{T}}$ and $c_0 = [0, 0, 0]^{\mathrm{T}}$.

Suppose that the H_∞ performance index $\gamma = 0.8$ and the delay $d(t) = 0.1 \sin t$ (so $\tau = \mu = 0.1$). By Theorem 1 and Algorithm 1, we can figure out the feedback matrix $\boldsymbol{K}$ as follows:

$$\begin{bmatrix} 1.4296 & 0.9149 & -0.6213 & 0.1279 & 0.2114 & 0.8991 \\ 0.2846 & 1.4329 & -1.0665 & -0.9176 & 0.1322 & 0.1044 \\ -1.0628 & -0.6188 & 2.0203 & 0.1012 & 0.8966 & -0.2231 \end{bmatrix}.$$

On one hand, Fig. 2 shows the position trajectories of all agents. It is clear that all agents reach an agreement on their positions while surrounding a common point c with a desired angular velocity ω on a plane perpendicular to the vector i_ω.

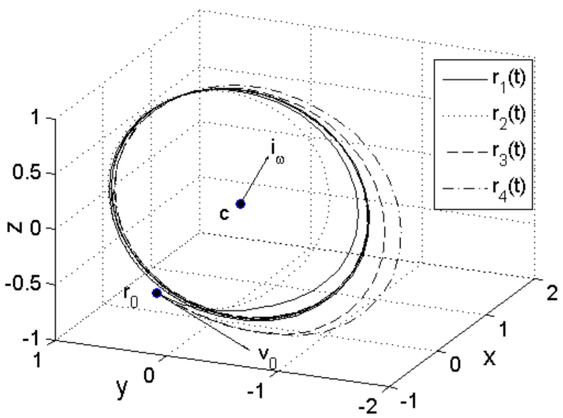

Fig. 2. Position trajectories of all agents

On the other hand, Fig. 3 shows the energy relation of the controlled output and external disturbance. Obviously, the rotating consensus of the multi-agent system is achieved with the H_∞ disturbance attenuation index.

Therefore, using the distributed protocol (3) and calculating the feedback ma-

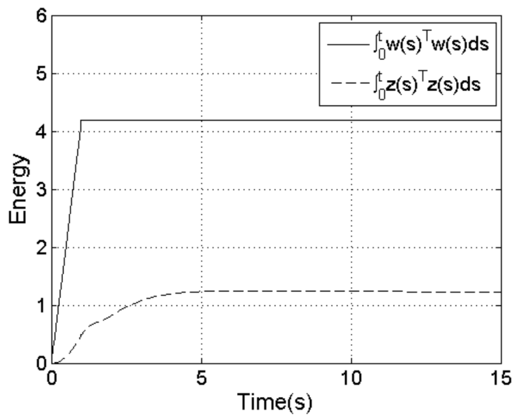

Fig. 3. Energy trajectories of the controlled output $z(t)$ and external disturbance $w(t)$

trix $\boldsymbol{K}$ by Theorem 1 and Algorithm 1, the multi-agent system can reach rotating consensus while satisfying the desired H_∞ disturbance attenuation index. So we validate the effectiveness of the proposed protocol and demonstrate the correctness of Theorem 1 and Algorithm 1.

5. Conclusion

In this paper, robust delay-dependent H_∞ control problems are studied for rotating consensus of second-order multi-agent systems with parameter uncertainties, external disturbances and time-varying delay in three-dimensional space. Firstly, a rotating consensus is defined in three-dimensional space. Based on state feedback of neighbors, a distributed control protocol is designed. Then a sufficient delay-dependent condition in terms of the matrix inequalities is derived to make all agents asymptotically reach rotating consensus with the desired H_∞ performance index. Furthermore, an algorithm is elaborately designed to get feasible solution to this condition. The main contributions of this paper are as follows: First, the influence of parameter uncertainty, external disturbances and time-varying delay on rotating consensus are considered at the same time; Second, a delay-dependent control condition is derived, whose complexity is lower because the system is decoupled; Third, an algorithm is proposed for calculating parameters of control protocol.

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